Truthmakers and Information States

Inclusion, Containment, Duality

Søren Brinck Knudstorp

February 19, 2026

LIRa Seminar

Plan for the talk

I'll discuss a cluster of observations on points of contact between truthmaker and information semantics. These fall under three connected themes:

- Information states, à la BSML, and Containment.
- Truthmakers and Inclusion.
- Translations.

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(Finean) Truthmaker Semantics

Definition (Semantics)

Frames are complete posets $(S, \sqsubseteq)$.
The semantics is bilateral (truthmaking $\Vdash^+$ and falsitymaking $\Vdash^-$), and models come with two valuations $V^+, V^- : \mathbf{At} \to \mathcal{P}(S)$.

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\begin{array}{lll}
s \Vdash p & \text{iff} & s \in V^+(p) \\
s \dashv\mid p & \text{iff} & s \in V^-(p) \\
s \Vdash \neg\varphi & \text{iff} & s \dashv\mid \varphi \\
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$s \dashv\vert p$	iff	$s \in V^{-}(p)$
$s \Vdash \neg\varphi$	iff	$s \dashv\vert \varphi$
$s \dashv\vert \neg\varphi$	iff	$s \Vdash \varphi$
$s \Vdash \varphi \vee \psi$	iff	$s \Vdash \varphi$ or $s \Vdash \psi$

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$s \dashv\!\mid \neg\varphi$	iff	$s \Vdash \varphi$
$s \Vdash \varphi \vee \psi$	iff	$s \Vdash \varphi$ or $s \Vdash \psi$
$s \dashv\!\mid \varphi \vee \psi$	iff	$\exists\, s', s''$ such that $s' \dashv\!\mid \varphi; s'' \dashv\!\mid \psi;$ and $s = s' \sqcup s''$
$s \Vdash \varphi \wedge \psi$	iff	$\exists\, s', s''$ such that $s' \Vdash \varphi; s'' \Vdash \psi;$ and $s = s' \sqcup s''$

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s \dashv\!\mid \varphi \vee \psi & \text{iff} & \exists\, s', s'' \text{ such that } s' \dashv\!\mid \varphi; s'' \dashv\!\mid \psi; \text{ and } s = s' \sqcup s'' \\
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s \dashv\!\mid \varphi \wedge \psi & \text{iff} & s \Vdash \varphi \text{ or } s \Vdash \psi.
\end{array}
$$

Many more design choices, including:
Inclusive disjunction. $s \Vdash \varphi \vee \psi$ iff $s \Vdash \varphi$ or $s \Vdash \psi$ or $s \Vdash \varphi \wedge \psi$

Inferential patterns:

Inferential patterns:

$$p \Vdash p \vee q$$

$$p \wedge q \nVdash p$$

Bilateral State-based Modal Logic (BSML) [Aloni (2022)]

Traditionally (in, e.g., CPC), formulas φ are evaluated at single valuations $v : \mathbf{At} \to \{0, 1\}$, $v \vDash \varphi$.

In BSML, like in inquisitive semantics, formulas are evaluated at sets of valuations ('teams') $t \subseteq \{v \mid v : \mathbf{At} \to \{0, 1\}\}$, $t \vDash \varphi$.

Definition (Semantic clauses)

For $t \subseteq \{v \mid v : \mathbf{At} \to \{0, 1\}\}$, we define

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Observation 2: Telltale of containment logics

Two guiding themes:

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1. Points of contact between BSML and truthmaker semantics.

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1. Points of contact between BSML and truthmaker semantics.
2. BSML-style information semantics for containment logics.

Semantics for containment logics.

Containment and relevance

Containment logics obey the proscriptive principle:

$$\varphi \vdash \psi \qquad \text{implies} \qquad \mathbf{At}(\varphi) \supseteq \mathbf{At}(\psi).$$

Strong form of variable sharing:

$$\varphi \vdash \psi \qquad \text{implies} \qquad \mathbf{At}(\varphi) \cap \mathbf{At}(\psi) \neq \varnothing.$$

Signature invalidities:

1. $p \wedge \neg p \nvdash q$ [like relevant logics]
2. $p \nvdash q \vee \neg q$ [like relevant logics]
3. $p \nvdash p \vee q$ [like BSML]

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Angell's Analytic Entailment (AC)

One prominent containment logic is Angell's analytic entailment AC.
AC is, as shown by Ferguson (2016) and Fine (2016), the containment fragment of FDE:

$$\varphi \vdash_{AC} \psi \qquad \text{iff} \qquad \varphi \vdash_{FDE} \psi \text{ and } \mathbf{Lit}(\varphi) \supseteq \mathbf{Lit}(\psi).$$

Of interest to us because:

- It is a containment logic.
- Fine (2016) provided a complete truthmaker semantics for AC.

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First goal: BSML-style semantics for AC.

BSML and classicality

Recall the BSML semantics: for $t \in \mathcal{P}(\{v \mid v : \mathbf{At} \to \{0,1\}\}$ we define

$t \vDash p$	**iff**	for all $v \in t, v(p) = 1$
$t \dashv p$	**iff**	for all $v \in t, v(p) = 0$
$t \vDash \neg\varphi$	**iff**	$t \dashv \varphi$
$t \dashv \neg\varphi$	**iff**	$t \vDash \varphi$
$t \vDash \varphi \vee \psi$	**iff**	$\exists\, t', t''$ such that $t' \vDash \varphi; t'' \vDash \psi;$ and $t = t' \cup t''$
$t \dashv \varphi \vee \psi$	**iff**	$t \dashv \varphi$ and $t \dashv \psi$
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$t \models p$	iff	for all $v \in t, v(p) = 1$
$t \mathrel{=\!\!|} p$	iff	for all $v \in t, v(p) = 0$
$t \models \neg\varphi$	iff	$t \mathrel{=\!\!|} \varphi$
$t \mathrel{=\!\!|} \neg\varphi$	iff	$t \models \varphi$
$t \models \varphi \vee \psi$	iff	$\exists\, t', t''$ such that $t' \models \varphi; t'' \models \psi;$ and $t = t' \cup t''$
$t \mathrel{=\!\!|} \varphi \vee \psi$	iff	$t \mathrel{=\!\!|} \varphi$ and $t \mathrel{=\!\!|} \psi$
$t \models \varphi \wedge \psi$	iff	$t \models \varphi$ and $t \models \psi$
$t \mathrel{=\!\!|} \varphi \wedge \psi$	iff	$\exists\, t', t''$ such that $t' \mathrel{=\!\!|} \varphi; t'' \mathrel{=\!\!|} \psi;$ and $t = t' \cup t''$.

Problem: $p \wedge \neg p \models q$.

BSML and classicality

Four-valued BSML semantics: for $t \in \mathcal{P}(\{v \mid v : \mathbf{At} \to \mathcal{P}(\{0,1\})\})$ we define

$t \vDash p$	iff	for all $v \in t, v(p) \ni 1$
$t \dashv p$	iff	for all $v \in t, v(p) \ni 0$
$t \vDash \neg\varphi$	iff	$t \dashv \varphi$
$t \dashv \neg\varphi$	iff	$t \vDash \varphi$
$t \vDash \varphi \vee \psi$	iff	$\exists\, t', t''$ such that $t' \vDash \varphi; t'' \vDash \psi;$ and $t = t' \cup t''$
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$t \vDash \varphi \wedge \psi$	iff	$t \vDash \varphi$ and $t \vDash \psi$
$t \dashv \varphi \wedge \psi$	iff	$\exists\, t', t''$ such that $t' \dashv \varphi; t'' \dashv \psi;$ and $t = t' \cup t''$.

Problem solved: $p \wedge \neg p \nvDash q$. ✓

BSML-style semantics for AC

FDE semantics: Given $\mathcal{P}(X)$, $V^+, V^- : \mathbf{At} \to \mathcal{P}\mathcal{P}(X)$ s.t.

- $V^+(p)$ is a non-empty ideal;
- $V^-(p)$ is a non-empty ideal,

we define for $t \in \mathcal{P}(X)$

$t \vDash p$	iff	$t \in V^+(p)$
$t \dashv p$	iff	$t \in V^-(p)$
$t \vDash \neg\varphi$	iff	$t \dashv \varphi$
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$t \dashv \varphi \wedge \psi$	iff	$\exists t', t''$ such that $t' \dashv \varphi; t'' \dashv \psi;$ and $t = t' \cup t''$.

Theorem (FDE completeness)

$\varphi \vDash \psi$ *if and only if* $\varphi \vdash_{FDE} \psi$.

BSML-style semantics for AC

AC semantics: Given $\mathcal{P}(X)$, $V^+, V^- : \mathbf{At} \to \mathcal{P}\mathcal{P}(X)$ s.t.

- $V^+(p)$ is an ideal;
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we define for $t \in \mathcal{P}(X)$

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$t \mathrel{=\!\!\mid} p$	iff	$t \in V^-(p)$
$t \vDash \neg\varphi$	iff	$t \mathrel{=\!\!\mid} \varphi$
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Theorem (AC completeness)

$\varphi \vDash \psi$ *if and only if* $\varphi \vdash_{AC} \psi$.

BSML-style semantics for AC

Four-val. BSML* semantics: Given $\mathcal{P}(X)$, $V^+, V^- : \mathbf{At} \to \mathcal{PP}(X)$ s.t.

- $V^+(p)$ is an ideal but for the empty set;
- $V^-(p)$ is an ideal but for the empty set,

we define for $t \in \mathcal{P}(X)$

$t \vDash p$	iff	$t \in V^+(p)$
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Theorem (Four-val. BSML* completeness)

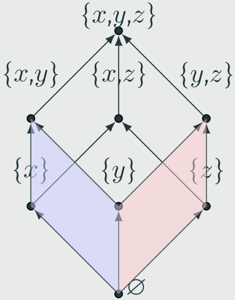
$\varphi \vDash \psi$ *if and only if* $\varphi \vDash_{BSML*} \psi$.

FDE, AC, and BSML*

FDE

Always: $V^{\pm}(p) = \mathcal{I} \ni \varnothing$.

Example:



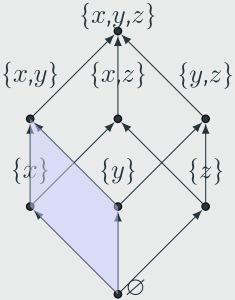
$V^{+}(p) = blue$;

$V^{-}(p) = red$.

AC

Possibly: $V^{\pm}(p) = \mathcal{I}$.

Example:



$V^{+}(p) = blue$;

$V^{-}(p) = red$.

BSML*

Never: $V^{\pm}(p) = \mathcal{I} \not\ni \varnothing$.

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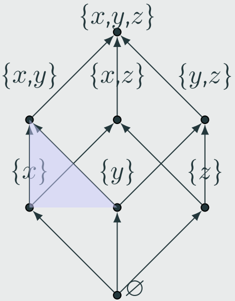

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- Of other team logics?
- More syntactic characterization(s)?

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- What other containment logics arise by varying the frames (lattices, semilattices, distributive semilattices, etc.) or valuations?

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Follow-ups:

- What other containment logics arise by varying the frames (lattices, semilattices, distributive semilattices, etc.) or valuations?
- For instance, can we obtain a complete semantics for Correia's (2016) logic of factual equivalence?

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A semantics deserves an informal conceptual gloss

Interpreting the semantics for Analytic Containment

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Say that:

x contains the information that p if there is $x' \sqsubseteq x$ s.t. $x' \in V^+(p)$.

I.e.

$$x \models p \qquad \text{iff} \qquad \exists x' \sqsubseteq x \text{ such that } x' \in V^+(p).$$

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Say that:

x contains the information that $\varphi \wedge \psi$ if it contains the information that φ and it contains the information that ψ.

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Say that:

x contains the information that $\varphi \vee \psi$ if there are states x' and x'' containing the information that φ and that ψ, respectively, such that $x = x' \sqcap x''$.

I.e.

$$x \vDash \varphi \vee \psi \quad \textbf{iff} \quad \exists\, x', x'' \text{ such that } x' \vDash \varphi; x'' \vDash \psi; \text{ and } x = x' \sqcap x''.$$

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Say that:

x contains the information that $\neg\varphi$ if it contains the information that not φ, defined dually.

I.e.

$$x \models \neg\varphi \qquad \text{iff} \qquad x \dashv \varphi.$$

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In sum:

$x \models p$	iff	$\exists x' \sqsubseteq x$ such that $x' \in V^+(p)$,
$x \dashv p$	iff	$\exists x' \sqsubseteq x$ such that $x' \in V^-(p)$,
$x \models \neg\varphi$	iff	$x \dashv \varphi$,
$x \dashv \neg\varphi$	iff	$x \models \varphi$,
$x \models \varphi \wedge \psi$	iff	$x \models \varphi$ and $x \models \psi$,
$x \dashv \varphi \wedge \psi$	iff	$\exists\, x', x''$ such that $x' \dashv \varphi; x'' \dashv \psi;$ and $x = x' \sqcap x''$,
$x \models \varphi \vee \psi$	iff	$\exists\, x', x''$ such that $x' \models \varphi; x'' \models \psi;$ and $x = x' \sqcap x''$,
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$x \vDash p$	iff	$\exists x' \sqsubseteq x$ such that $x' \in V^+(p)$,
$x \Dashv p$	iff	$\exists x' \sqsubseteq x$ such that $x' \in V^-(p)$,
$x \vDash \neg\varphi$	iff	$x \Dashv \varphi$,
$x \Dashv \neg\varphi$	iff	$x \vDash \varphi$,
$x \vDash \varphi \wedge \psi$	iff	$x \vDash \varphi$ and $x \vDash \psi$,
$x \Dashv \varphi \wedge \psi$	iff	$\exists\, x', x''$ such that $x' \Dashv \varphi; x'' \Dashv \psi;$ and $x = x' \sqcap x''$,
$x \vDash \varphi \vee \psi$	iff	$\exists\, x', x''$ such that $x' \vDash \varphi; x'' \vDash \psi;$ and $x = x' \sqcap x''$,
$x \Dashv \varphi \vee \psi$	iff	$x \Dashv \varphi$ and $x \Dashv \psi$.

Then: $\varphi \vdash_{AC} \psi$ if and only if $\varphi \vDash \psi$.

Recall

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Inferential patterns:

$$p \not\models p \vee q$$

$$p \wedge q \models p$$

Observation 1: Mirror image of truthmaker entailment

Observation 2: Telltale of containment logics

And recall the two guiding themes:

1. Points of contact between BSML and truthmaker semantics.
2. BSML-style semantics for containment logics. ✓

Truthmakers and Inclusion.

Replete truthmaker entailment

Write $\varphi \Vdash \psi$ for replete truthmaker preservation.

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□

Theorem[3]

Replete truthmaker entailment is the inclusion fragment of FDE; i.e.,

$$\varphi \Vdash \psi \qquad \text{iff} \qquad \varphi \vdash_{FDE} \psi \text{ and } \mathbf{Lit}(\varphi) \subseteq \mathbf{Lit}(\psi).$$

[3] I imagine this is known, but I haven't found it stated.

A sample of corollaries

Corollary

$$\varphi \vdash_{FDE} \psi \text{ and } \mathbf{Lit}(\varphi) = \mathbf{Lit}(\psi) \quad \text{iff} \quad \varphi \vDash \psi \text{ and } \neg\psi \vDash \neg\varphi$$
$$\text{iff} \quad \neg\psi \Vdash \neg\varphi \text{ and } \varphi \Vdash \psi.$$

$$\varphi \dashv\vdash_{FDE} \psi \text{ and } \mathbf{Lit}(\varphi) \supseteq \mathbf{Lit}(\psi) \quad \text{iff} \quad \varphi \vDash \psi \text{ and } \neg\varphi \vDash \neg\psi$$
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Corollary (when $V^+(p) \neq \varnothing \Leftrightarrow V^-(p) \neq \varnothing$)

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Before we proceed, two further remarks on truthmakers and inclusion.

Maxim: *Exactify!*

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Remark 1: On what it means for a semantics to be *exact*.

When is a semantics *exact*?

- Say that $\models$ satisfies the inclusion principle if

$$\varphi \models \psi \qquad \text{implies} \qquad \mathbf{At}(\varphi) \subseteq \mathbf{At}(\psi).$$

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– Caveat 1: $\varphi \wedge (\varphi \to \psi) \Vdash \psi$ only when $\mathbf{At}(\varphi) \subseteq \mathbf{At}(\psi)$?[4]

– Caveat 2: How about explosion and its dual? Perhaps inclusion *modulo* explosion and its dual?[5]

[4]Consider, e.g., exact preservation for the intuitionistic semantics; does this hold there? I've checked that $(p \wedge q) \wedge (p \wedge q \to p) \nVdash p$, as we would like. Do we have $\varphi \wedge (\varphi \to \psi) \models \varphi \wedge \psi$? And what about $(p \to q) \wedge (q \to r) \Vdash p \to r$?

[5]The signature invalidities of 'inclusion logics' include explosion and its dual, but maybe exactness should only generalize the invalidity of simplification (think counterfactuals, modalities, etc.).

Remark 2: On relevance and wholly relevance.

A-B Analysis: Relevance and Wholly Relevance

Recall Anderson and Belnap's (1962) tautological entailments:

1) For A_i a conjunction of literals, and B_j a disjunction of literals, let

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[6] AC can be given a similar A-B analysis.

Follow-ups and future work

Follow-ups I'd like to think about:

1. Like replete entailment, can other truthmaker entailments be given a double-barreled analysis?
2. For instance, can (non-)inclusive entailment be captured by stronger inclusion principles?
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Translations.

Source logic: BSML with NE and ⧫

Fix a finite set of propositional variables $\mathbf{At}$, and define:

$$\varphi ::= \bot \mid \text{NE} \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \blacklozenge\varphi.$$

Definition

For $t \subseteq \{v \mid v : \mathbf{At} \to \{0,1\}\}$, we have

$t \models \text{NE}$	iff	$t \neq \varnothing$
$t \dashv \text{NE}$	iff	$t = \varnothing$
$t \models \blacklozenge\varphi$	iff	$\exists\, s \subseteq t$ such that $\varnothing \neq s \models \varphi$
$t \dashv \blacklozenge\varphi$	iff	$\forall s \subseteq t$: $s \dashv \varphi$
$t \models \bot$	iff	$t = \varnothing$
$t \dashv \bot$	always	

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Target logic: modal information logic

Target logic is the modal logic in the language with two modalities,

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle \sup \rangle \varphi\varphi \mid \langle \mathsf{s}^* \rangle \varphi,$$

for $p \in \mathbf{At}_{\pm} := \{p_+, p_- \mid p \in \mathbf{At}\}$, interpreted over distributive semilattices $(S, \curlyvee)$, where

$s \Vdash \langle \sup \rangle \varphi\psi$ iff $\exists\, t, u$ s.t. $t \Vdash \varphi,\ u \Vdash \psi,$ and $s = t \curlyvee u.$

$s \Vdash \langle \mathsf{s}^* \rangle \varphi$ iff $\exists\, s_1, \ldots, s_n$ s.t. each $s_i \Vdash \varphi$ and $s = s_1 \curlyvee \cdots \curlyvee s_n.$

Objective: Define translation pair $\cdot^+, \cdot^-$ s.t. for all φ, ψ:

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Translating BSML

Set

$$\Gamma := \{\mathsf{H}(\mathsf{NE}^+ \vee \mathsf{NE}^-), \bigwedge_{p \in \mathbf{At}} \langle \sup \rangle p^+ p^-\},$$

and define $\cdot^+, \cdot^-$ via the double-recursive clauses:

$$\begin{array}{rcl} \bot^+ & := & \mathsf{NE}^- \\ \mathsf{NE}^+ & := & \bigwedge_{p \in \mathbf{At}} \neg(p^+ \wedge p^-) \\ p^+ & := & \mathsf{H}\langle \mathsf{s}^* \rangle p_+ \\ (\neg\varphi)^+ & := & \varphi^- \\ (\varphi \vee \psi)^+ & := & \langle \sup \rangle \varphi^+ \psi^+ \\ (\varphi \wedge \psi)^+ & := & \varphi^+ \wedge \psi^+ \\ (\blacklozenge\varphi)^+ & := & \mathsf{P}(\mathsf{NE}^+ \wedge \varphi^+) \end{array} \qquad \begin{array}{rcl} \bot^- & := & \top \\ \mathsf{NE}^- & := & \bigwedge_{p \in \mathbf{At}} (p^+ \wedge p^-) \\ p^- & := & \mathsf{H}\langle \mathsf{s}^* \rangle p_- \\ (\neg\varphi)^- & := & \varphi^+ \\ (\varphi \vee \psi)^- & := & \varphi^- \wedge \psi^- \\ (\varphi \wedge \psi)^- & := & \langle \sup \rangle \varphi^- \psi^- \\ (\blacklozenge\varphi)^- & := & \mathsf{H}\varphi^-. \end{array}$$

Theorem

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BSML translation contra truthmaker translation

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Translation clauses for truthmaker semantics:[8]

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Translating inquisitive logic

For the case of inquisitive logic, translate $\mathbb{v}$, $\to$ as follows:

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Theorem (translation of Inq)

$$\varphi \vDash \psi \qquad \text{iff} \qquad \Gamma, \varphi^+ \Vdash \psi^+.$$

Remark

The translation can be extended to other propositional team logics too, including all fragments of the grammar:

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Comments and follow-ups on translation

1. The translation isolates the structural features of powersets needed for 'team-semantical reasoning': we go from powersets to (mere) distributive semilattices. Translation as gateway to abstract (generalized) team semantics? To labelled calculi?
2. Team logics are decidable contra target modal logic.
3. Comparison with weak positive logic.[9]

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